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SPECTRAL PROBLEMS ARISING IN THE STABILIZATION PROBLEM FOR THE LOADED HEAT EQUATION: A TWO-DIMENSIONAL AND MULTI-POINT CASES

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Abstract. Spectral properties of a loaded two-dimensional Laplace operator, studied in this work are the application with the stabilization of solutions of problems for the heat equation. The stabilization problem (of forming a cylinder) of a solution of boundary value problem for heat equation with the loaded two-dimensional Laplace operator is considered. An algorithm is proposed for approximate construction of boundary controls providing the required stabilization of the solution. The work continues the research of the authors carried out earlier for the loaded one-dimensional heat equation.

The idea of reducing the stabilization problem for a parabolic equation by means of boundary controls to the solution of an auxiliary boundary value problem in the extended domain of independent variables belongs to A.V. Fursikov. At the same time, recently, the so-called loaded differential equations are actively used in problems of mathematical modeling and control of nonlocal dynamical systems.

Key words: boundary stabilization, heat equation, spectrum, loaded Laplace operator.

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1 Introduction

The idea of reducing the stabilization problem for a parabolic equation by means of boundary controls to an auxiliary boundary value problem in the extended domain of independent variables belongs to A.V. Fursikov. It was proposed in his work [1] and developed further in the works [2, 3, 4]. At the same time, recently, the so-called loaded differential equations [5, 6, 7, 8, 9, 10] are actively used in problems of mathematical modeling and control of nonlocal dynamical systems. We have previously studied stabilization problems for a loaded one- and two-dimensional heat equations [11, 12, 13]. In this work, we investigate the spectral properties of the loaded two-dimensional Laplace operator, which are applied to the solution of the stabilization problem.

Let $\Omega = \{x, y : -\pi/2 < x, y < \pi/2\}$ be a domain with a boundary $\partial\Omega$. In the cylinder $Q = \Omega \times \{t > 0\}$ with lateral surface $\Sigma = \partial\Omega \times \{t > 0\}$ we consider the boundary value problem for the loaded heat equation

$$u_t - \Delta u + \sum_{m=1}^{M} \alpha_m \cdot u(x_m, y, t) + \sum_{n=1}^{N} \beta_n \cdot u(x, y_n, t) = 0, \ \{x, y, t\} \in Q, \tag{1}$$

$$u(x,y,0)=u_0(x,y),\ \{x,y\}\in\Omega, \tag{2}$$

$$u(x,y,t)=p(x,y,t),\ \{x,y,t\}\in\Sigma, \tag{3}$$

where $\{x_m,\ y_n,\ m=1,...,M,\ n=1,...,N\}\subset(-\pi/2,\pi/2)$ are fixed, $\{\alpha_m,\ \beta_n,\ m=1,...,M,\ n=1,...,N\}\subset\mathbb{C}$ are given complex numbers, $u_0(x,y)\in L_2(\Omega)$ is given function.

The aim is to find a function $p(x,y,t)$ such that a solution of the boundary value problem satisfies the inequality

$$\|u(x,y,t)\|_{L_2(\Omega)}\le C_0e^{-\sigma t},\ \sigma>0,\ t>0. \tag{4}$$

Equation (1) is called a loaded equation [5, 6]. We note that problem (1)–(4) with a single load point was studied in [13].

For problem (1)–(4) in Section 2, an auxiliary stabilization problem is associated with it by expanding the region of independent variables. To solve this auxiliary problem (5)–(7) in Section 3 we consider spectral properties of the loaded two-dimensional Laplace operator. Section 4 contains the main results of the work which were formulated in the form of Lemmas 4.1, 4.2, 4.3 and 4.4, establishing the desired spectral properties. Sections 5 and 6 give the proofs of Lemmas 4.1 and 4.2, respectively. Sections 7 and 8 give the proofs of Lemmas 4.3 and 4.4, respectively. On the basis of these results, an algorithm for solving the stabilization problem (1)–(4) is proposed. In Section 9, an algorithm for solving problem (1)–(4) is given, where the solution of problem (1)–(4) is found as a trace of the solution of the auxiliary problem (5)–(7).

2 The auxiliary boundary value problem

We introduce the auxiliary boundary value problem. Let $\Omega_1=\{x,y:-\pi<x,y<\pi\}$ and $Q_1=\Omega_1\times\{t>0\}$.

$$z_t-\Delta z+\sum_{m=1}^{M}\alpha_m\cdot z(x_m,y,t)+\sum_{n=1}^{N}\beta_n\cdot z(x,y_n,t)=0,\ \{x,y,t\}\in Q_1, \tag{5}$$

$$z(x,y,0)=z_0(x,y),\ \{x,y\}\in\Omega_1, \tag{6}$$

$$\begin{aligned}&\frac{\partial^j z(-\pi,y,t)}{\partial x^j}=\frac{\partial^j z(\pi,y,t)}{\partial x^j},\ \{y,t\}\in(-\pi,\pi)\times\{t>0\},\\ &\frac{\partial^j z(x,-\pi,t)}{\partial y^j}=\frac{\partial^j z(x,\pi,t)}{\partial y^j},\ \{x,t\}\in(-\pi,\pi)\times\{t>0\},\end{aligned} \tag{7}$$

$j=0,1.$

The problem is to find an initial function $z_0(x,y)$ such that a solution of the BVP satisfies the inequality

$$\|z(x,y,t)\|_{L_2(\Omega_1)}\le C_0e^{-\sigma t},\ \sigma>0,\ t>0, \tag{8}$$

where the constants C_0 and σ are the same as in the original problem (1)–(4).

3 Statement of the spectral problems for the loaded two-dimensional Laplace operator

Let us search for a solution of the problem (5)–(7) in the form

$$z(x,y,t)=\sum_{k,l\in \mathrm{Z}} Z_{kl}(t)\psi_{kl}(x,y),\ \ Z_{kl}(t)=\left(z(x,y,t),\varphi_{kl}(x,y)\right), \tag{9}$$

where $(\cdot,\cdot)$ denotes a scalar product, and $\{\varphi_{kl}(x,y),\ k,l\in \mathrm{Z}\}$, $\{\psi_{kl}(x,y),\ k,l\in \mathrm{Z}\}$ are a biorthogonal bases of the space $L_2((-\pi,\pi)^2)$ and $\mathrm{Z}=\{0,\pm1,\pm2,...\}$. The following two spectral problems are considered for construction of the biorthogonal bases $\{\varphi_{kl}(x,y),\ \ k,l\in \mathrm{Z}\}$, $\{\psi_{kl}(x,y),\ k,l\in \mathrm{Z}\}$ in the domain $Q=\{x,y: -\pi<x<\pi, -\pi<y<\pi\}$:

$$\begin{cases} -\Delta\varphi(x,y)+\sum\limits_{m=1}^{M}\alpha_m\varphi(x_m,y)=\lambda\varphi(x,y), \\[2mm] \dfrac{\partial^j\varphi(-\pi,y)}{\partial x^j}=\dfrac{\partial^j\varphi(\pi,y)}{\partial x^j},\ \ \dfrac{\partial^j\varphi(x,-\pi)}{\partial y^j}=\dfrac{\partial^j\varphi(x,\pi)}{\partial y^j}, \end{cases} \tag{10}$$

$$\begin{cases} -\Delta\varphi(x,y)+\sum\limits_{m=1}^{M}\alpha_m\varphi(x_m,y)+\sum\limits_{n=1}^{N}\beta_n\varphi(x,y_n)=\lambda\varphi(x,y), \\[2mm] \dfrac{\partial^j\varphi(-\pi,y)}{\partial x^j}=\dfrac{\partial^j\varphi(\pi,y)}{\partial x^j},\ \ \dfrac{\partial^j\varphi(x,-\pi)}{\partial y^j}=\dfrac{\partial^j\varphi(x,\pi)}{\partial y^j}, \end{cases} \tag{11}$$

where $j=0,1$, Δ is the Laplace operator, $\{\alpha_m,\ \beta_n,\ m=1,...,M,\ n=1,...,N\}\subset\mathbb{C}$ are given complex numbers, $\lambda\in\mathbb{C}$ is a spectral parameter.

4 Main results

Let $\mathrm{Z}=\{0,\pm1,\pm2,...\}$, $\bar\alpha=\sum\limits_{m=1}^{M}\alpha_m$, $\bar\beta=\sum\limits_{n=1}^{N}\beta_n$, $\bar x=\sum\limits_{m=1}^{M}x_m$, $\bar y=\sum\limits_{n=1}^{N}y_n$. The following propositions are valid.

Lemma 4.1. *(a). Let $\forall\ l\in \mathrm{Z}:\bar\alpha\neq l^2$. Then a system of eigenfunctions and eigenvalues of the problem (10) is defined in the form:*

$$\left\{\begin{aligned} &\varphi_{kl}(x,y)=\left(e^{ilx}+\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{ilx_m}}{l^2-\bar\alpha}\right)e^{iky},\ \lambda_{kl}=l^2+k^2,\ l\in \mathrm{Z}'\equiv \mathrm{Z}\backslash\{0\};\\ &\varphi_{k0}(x,y)=e^{iky},\ \lambda_{k0}=\bar\alpha+k^2\ (l=0),\ k\in \mathrm{Z}\end{aligned}\right\}. \tag{12}$$

(b). Let $\exists\ l_0\in \mathrm{Z}:\ \bar\alpha=l_0^2$. Then a system of eigenfunctions, associated functions (marked with $\sim$) and eigenvalues of the problem (10) is defined in the form:

$$\left\{\varphi_{kl}(x,y)=\left(e^{ilx}+\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{ilx_m}}{l^2-\bar\alpha}\right)e^{iky},\ \lambda_{kl}=l^2+k^2,\ l\in \mathrm{Z}_1'\equiv \mathrm{Z}'\backslash\{\pm l_0\};\right.$$

$$\left.\varphi_{kl_0}(x,y)=e^{iky},\ \tilde{\varphi}^{\pm}_{kl_0}(x,y)=e^{\pm il_0(x-\bar{x})+iky},\ \lambda_{kl_0}=\bar{\alpha}+k^2\ \left(\bar{\alpha}=l_0^2\right),\ k\in \mathrm{Z}\right\}. \quad (13)$$

Lemma 4.2. *(a). Let $\forall\ l\in \mathrm{Z}:\ \bar{\alpha}\neq l^2$. Then a biorthogonal sequence for the basis (12) is*

$$\{\psi_{kl}(x,y),\ k,l\in \mathrm{Z}\}=$$

$$=\left\{e^{i(lx+ky)},\ -\frac{1}{2\pi}\sum_{r\in \mathrm{Z}}\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{i(r(x+x_m)+ky)}}{r^2-\bar{\alpha}},\ l\in \mathrm{Z}',\ k\in \mathrm{Z}\right\}, \quad (14)$$

which defines a biorthogonal basis in $L_2((-\pi,\pi)\times(-\pi,\pi))$.

(b). Let $\exists\ l_0\in \mathrm{Z}:\ \bar{\alpha}=l_0^2$. Then a biorthogonal sequence for the basis (13) is

$$\{\psi_{kl}(x),\ k,\ l\in \mathrm{Z}\}=$$

$$=\left\{e^{i(lx+ky)},\ -\frac{1}{2\pi}\sum_{r\in \mathrm{Z}\setminus\{\pm l_0\}}\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{i(r(x+x_m)+ky)}}{r^2-\bar{\alpha}},\ k\in \mathrm{Z},l\in \mathrm{Z}'\right\}. \quad (15)$$

which defines a biorthogonal basis in $L_2((-\pi,\pi)\times(-\pi,\pi))$.

Lemma 4.3. *(a). Let $\forall\ k,\ l\in \mathrm{Z}:\ \bar{\beta}\neq k^2,\ \bar{\alpha}\neq l^2$. Then a system of eigenfunctions and eigenvalues for the problem (11) is defined in the form:*

$$\left\{\varphi_{kl}(x,y)=\left(e^{ilx}+\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{ilx_m}}{l^2-\bar{\alpha}}\right)\left(e^{iky}+\frac{\sum\limits_{n=1}^{N}\beta_n\cdot e^{iky_n}}{k^2-\bar{\beta}}\right),\right.$$

$$\lambda_{kl}=k^2+l^2,\ k,l\in \mathrm{Z}';\ e^{ilx}+\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{ilx_m}}{l^2-\bar{\alpha}},\ \lambda_{0l}=\bar{\beta}+l^2,\ l\in \mathrm{Z}';$$

$$\left. e^{iky}+\frac{\sum\limits_{n=1}^{N}\beta_n\cdot e^{iky_n}}{k^2-\bar{\beta}},\ \lambda_{k0}=k^2+\bar{\alpha},\ k\in \mathrm{Z}';\ 1,\ \lambda_{00}=\bar{\alpha}+\bar{\beta}\right\}. \quad (16)$$

(b). Let $\forall\ k\in \mathrm{Z}:\ \bar{\beta}\neq k^2$ and $\exists\ l_0\in \mathrm{Z}:\ \bar{\alpha}=l_0^2$. Then a system of eigenfunctions, associated functions (marked with $\sim$) and eigenvalues for the problem (11) is defined in the form (where $\mathrm{Z}_1'=\mathrm{Z}'\setminus\{\pm l_0\}$):

$$\left\{\varphi_{kl}(x,y)=\left(e^{ilx}+\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{ilx_m}}{l^2-\bar{\alpha}}\right)\left(e^{iky}+\frac{\sum\limits_{n=1}^{N}\beta_n\cdot e^{iky_n}}{k^2-\bar{\beta}}\right),\right.$$

$$\lambda_{kl}=k^2+l^2,\ k\in \mathrm{Z}',\ l\in \mathrm{Z}_1';\ \varphi_{kl_0}(x,y)=e^{iky}+\frac{\sum\limits_{n=1}^{N}\beta_n\cdot e^{iky_n}}{k^2-\bar{\beta}},$$

$$\tilde{\varphi}^{\pm}_{kl_0}(x,y) = e^{\pm i l_0 (x-\bar{x})}\left(e^{iky} + \frac{\sum\limits_{n=1}^{N} \beta_n \cdot e^{iky_n}}{k^2-\bar{\beta}}\right),\ \lambda_{kl_0} = k^2+\bar{\alpha},\ \bar{\alpha} = l_0^2,\ k \in \mathrm{Z}';$$

$$\left.\varphi_{0l_0}(x,y) = 1,\ \tilde{\varphi}^{\pm}_{0l_0}(x,y) = e^{\pm i l_0 (x-\bar{x})},\ \lambda_{0l_0} = \bar{\alpha}+\bar{\beta}\right\}. \tag{17}$$

(c). Let $\forall\ l \in \mathrm{Z}:\ \bar{\alpha} \neq l^2$ and $\exists\ k_0 \in \mathrm{Z}:\ \bar{\beta} = k_0^2$. Then a system of eigenfunctions, associated (marked with $\sim$) functions and eigenvalues for the problem (11) is defined in the form (where $\mathrm{Z}'_2 = \mathrm{Z}' \backslash \{\pm k_0\}$):

$$\left\{\varphi_{kl}(x,y) = \left(e^{ilx} + \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{ilx_m}}{l^2-\bar{\alpha}}\right)\left(e^{iky} + \frac{\sum\limits_{n=1}^{N} \beta_n \cdot e^{iky_n}}{k^2-\bar{\beta}}\right),\right.$$

$$\lambda_{kl} = k^2+l^2,\ k \in \mathrm{Z}'_2,\ l \in \mathrm{Z}';\ \varphi_{k_0 l}(x,y) = e^{ilx} + \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{ilx_m}}{l^2-\bar{\alpha}},$$

$$\tilde{\varphi}^{\pm}_{k_0 l}(x,y) = e^{\pm i k_0 (y-\bar{y})}\left(e^{ilx} + \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{ilx_m}}{l^2-\bar{\alpha}}\right),\ \lambda_{k_0 l} = \bar{\beta}+l^2,\ \bar{\beta} = k_0^2,\ l \in \mathrm{Z}';$$

$$\left.\varphi_{k_0 0}(x,y) = 1,\ \tilde{\varphi}_{k_0 0}(x,y) = e^{\pm i k_0 (y-\bar{y})},\ \lambda_{k_0 0} = \bar{\alpha}+\bar{\beta}\right\}. \tag{18}$$

(d). Let $\exists\ k_0,\ l_0 \in \mathrm{Z}:\ \bar{\beta} = k_0^2,\ \bar{\alpha} = l_0^2$. Then a system of eigenfunctions, associated functions (marked with $\sim$) and eigenvalues for the problem (11) is defined in the form:

$$\left\{\varphi_{kl}(x,y) = \left(e^{ilx} + \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{ilx_m}}{l^2-\bar{\alpha}}\right)\left(e^{iky} + \frac{\sum\limits_{n=1}^{N} \beta_n \cdot e^{iky_n}}{k^2-\bar{\beta}}\right),\right.$$

$$\lambda_{kl} = k^2+l^2,\ k \in \mathrm{Z}'_2,\ l \in \mathrm{Z}'_1;\ \varphi_{k_0 l}(x,y) = e^{ilx} + \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{ilx_m}}{l^2-\bar{\alpha}},$$

$$\tilde{\varphi}^{\pm}_{k_0 l}(x,y) = e^{\pm i k_0 (y-\bar{y})}\left(e^{ilx} + \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{ilx_m}}{l^2-\bar{\alpha}}\right),\ \lambda_{k_0 l} = \bar{\beta}+l^2,\ \bar{\beta} = k_0^2,\ l \in \mathrm{Z}'_1;$$

$$\varphi_{kl_0}(x,y) = e^{iky} + \frac{\sum\limits_{n=1}^{N} \beta_n \cdot e^{iky_n}}{k^2-\bar{\beta}},\ \bar{\alpha}+k^2,\ k \in \mathrm{Z}'_2;$$

$$\varphi_{k_0 l_0}(x,y) = 1,\ \tilde{\varphi}_{k_0 l_0}(x,y) = e^{\pm i k_0 (y-\bar{y})},\ \lambda_{k_0 l_0} = \bar{\alpha}+\bar{\beta};$$

$$\tilde{\varphi}_{kl_0}(x,y) = e^{\pm il_0(x-\bar{x})}\left(e^{iky} + \frac{\sum\limits_{n=1}^{N}\beta_n\cdot e^{iky_n}}{k^2-\bar{\beta}}\right),\ \bar{\alpha}+k^2,\ k\in \mathrm{Z}_2';$$

$$\left.\tilde{\varphi}_{k_0l_0}(x,y) = e^{\pm il_0(x-\bar{x})},\ \tilde{\varphi}_{k_0l_0}(x,y) = e^{\pm il_0(x-\bar{x})\pm ik_0(y-\bar{y})},\ \lambda_{k_0l_0} = \bar{\alpha}+\bar{\beta}\right\}. \tag{19}$$

Lemma 4.4. *(a). Let $\forall\ k,\ l\in \mathrm{Z}:\ \bar{\beta}\neq k^2,\ \bar{\alpha}\neq l^2$. Then a biorthogonal sequence for the basis (16) is*

$$\{\psi_{kl}(x,y),\ k,l\in \mathrm{Z}\} = \left\{e^{i(lx+ky)},\ -\frac{1}{2\pi}\sum_{r\in \mathrm{Z}}\frac{\sum\limits_{n=1}^{N}\beta_n\cdot e^{i(r(y+y_n)+l(x+\bar{x}))}}{r^2-\bar{\beta}},\right.$$

$$-\frac{1}{2\pi}\sum_{r\in \mathrm{Z}}\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{i(r(x+x_m)+k(y+\bar{y}))}}{r^2-\bar{\alpha}},\ k,\ l\in \mathrm{Z}',$$

$$\left.\frac{1}{4\pi^2}\sum_{s,r\in \mathrm{Z}}\frac{\sum\limits_{m=1}^{M}\sum\limits_{n=1}^{N}\alpha_m\beta_n\cdot e^{i(r(x+x_m)+s(y+y_n))}}{\left(r^2-\bar{\alpha}\right)\left(s^2-\bar{\beta}\right)}\right\}, \tag{20}$$

which defines a biorthogonal basis in $L_2((-\pi,\pi)\times(-\pi,\pi))$.

(b). Let $\forall\ k\in \mathrm{Z}:\ \bar{\beta}\neq k^2$ and $\exists\ l_0\in \mathrm{Z}:\ \bar{\alpha}=l_0^2$. Then a biorthogonal sequence for the basis (17) is

$$\{\psi_{kl}(x),\ k,l\in \mathrm{Z}\} = \left\{e^{i(lx+ky)},\ -\frac{1}{2\pi}\sum_{r\in \mathrm{Z}}\frac{\sum\limits_{n=1}^{N}\beta_n\cdot e^{i(r(y+y_n)+l(x+\bar{x}))}}{r^2-\bar{\beta}},\right.$$

$$-\frac{1}{2\pi}\sum_{r\in \mathrm{Z}\setminus\{\pm l_0\}}\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{i(r(x+x_m)+k(y+\bar{y}))}}{r^2-\bar{\alpha}},\ k,\ l\in \mathrm{Z}',$$

$$\left.\frac{1}{4\pi^2}\sum_{r\in \mathrm{Z},\, s\in \mathrm{Z}\setminus\{\pm l_0\}}\frac{\sum\limits_{m=1}^{M}\sum\limits_{n=1}^{N}\alpha_m\beta_n\cdot e^{i(s(x+x_m)+r(y+y_n))}}{\left(s^2-\bar{\alpha}\right)\left(r^2-\bar{\beta}\right)}\right\}, \tag{21}$$

which defines a biorthogonal basis in $L_2((-\pi,\pi)\times(-\pi,\pi))$.

(c). Let $\forall\ l\in \mathrm{Z}:\ \bar{\alpha}\neq l^2$ and $\exists\ k_0\in \mathrm{Z}:\ \bar{\beta}=k_0^2$. Then a biorthogonal sequence for the basis (18) is

$$\{\psi_{kl}(x),\ k,l\in \mathrm{Z}\} = \left\{e^{i(lx+ky)}, -\frac{1}{2\pi}\sum_{r\in \mathrm{Z}\setminus\{\pm k_0\}}\frac{\sum\limits_{n=1}^{N}\beta_n\cdot e^{i(r(y+y_n)+l(x+\bar{x}))}}{r^2-\bar{\beta}},\right.$$

$$-\frac{1}{2\pi}\sum_{r\in \mathrm{Z}}\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{i(r(x+x_m)+k(y+\bar{y}))}}{r^2-\bar{\alpha}},\ k,\, l\in \mathrm{Z}',$$

$$\left.\frac{1}{4\pi^2}\sum_{r\in \mathrm{Z}\setminus\{\pm k_0\},\, s\in \mathrm{Z}}\frac{\sum\limits_{m=1}^{M}\sum\limits_{n=1}^{N}\alpha_m\beta_n\cdot e^{i(s(x+x_m)+r(y+y_n))}}{(s^2-\bar{\alpha})\left(r^2-\bar{\beta}\right)}\right\}, \tag{22}$$

which defines a biorthogonal basis in $L_2((-\pi,\pi)\times(-\pi,\pi))$.

(d). Let $\exists\ k_0,\, l_0\in \mathrm{Z}:\ \bar{\beta}=k_0^2,\ \bar{\alpha}=l_0^2$. *Then a biorthogonal sequence for the basis (19) is*

$$\{\psi_{kl}(x),\ k,l\in \mathrm{Z}\}=\left\{e^{i(lx+ky)},-\frac{1}{2\pi}\sum_{r\in \mathrm{Z}\setminus\{\pm k_0\}}\frac{\sum\limits_{n=1}^{N}\beta_n\cdot e^{i(r(y+y_n)+l(x+\bar{x}))}}{r^2-\bar{\beta}},\right.$$

$$-\frac{1}{2\pi}\sum_{r\in \mathrm{Z}\setminus\{\pm l_0\}}\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{i(r(x+x_m)+k(y+\bar{y}))}}{r^2-\bar{\alpha}},\ k,\, l\in \mathrm{Z}',$$

$$\left.\frac{1}{4\pi^2}\sum_{r\in \mathrm{Z}\setminus\{\pm k_0\},\, s\in \mathrm{Z}\setminus\{\pm l_0\}}\frac{\sum\limits_{m=1}^{M}\sum\limits_{n=1}^{N}\alpha_m\beta_n\cdot e^{i(s(x+x_m)+r(y+y_n))}}{(s^2-\bar{\alpha})\left(r^2-\bar{\beta}\right)}\right\}, \tag{23}$$

which defines a biorthogonal basis in $L_2((-\pi,\pi)\times(-\pi,\pi))$.

A one-dimensional analogue of the problems (10) and (11) is studied in [13].

5 Proof of Lemma 4.1

Using the method of separation of variables

$$\varphi_{kl}(x,y)=X_l(x)Y_k(y),\ k,\, l\in \mathrm{Z}, \tag{24}$$

$$\frac{-X_l''(x)+\sum\limits_{m=1}^{M}\alpha_m X_l(x_m)}{X_l(x)}+\frac{-Y_k''(y)}{Y_k(y)}=\lambda_{kl}\equiv\lambda_l^{(1)}+\lambda_k^{(2)},\ k,\, l\in \mathrm{Z}, \tag{25}$$

and for the solution of (10) we obtain the following spectral problems

$$\begin{cases} -X_l''(x)+\sum\limits_{m=1}^{M}\alpha_m X_l(x_m)=\lambda_l^{(1)}X_l(x), \\ X_l^{(j)}(-\pi)=X_l^{(j)}(\pi),\quad j=0,1,\, l\in \mathrm{Z}, \end{cases} \tag{26}$$

$$\begin{cases} -Y_k''(y)=\lambda_k^{(2)}Y_k(y), \\ Y_k^{(j)}(-\pi)=Y_k^{(j)}(\pi),\quad j=0,1,\, k\in \mathrm{Z}. \end{cases} \tag{27}$$

As we know, the solution of problem (27) has the form:

$$\left\{Y_k(y)=e^{i\,k\,y},\ \lambda_k^{(2)}=k^2,\ k\in \mathrm{Z}\right\}. \tag{28}$$

For the problem (26) it is necessary to consider the following two cases: (a) $\nexists\, l\in \mathrm{Z}:\ \bar{\alpha}=l^2$; (b) $\exists\, l_0\in \mathrm{Z}:\ \bar{\alpha}=l_0^2$.

(a) Let $\nexists\, l\in \mathrm{Z}:\ \bar{\alpha}=l^2$. We note that the general solution of the loaded differential equation (26) is represented as a linear combination of the following complete system of periodic functions:

$$\left\{\Phi_l(x)=e^{i\,l\,x},\ l\in \mathrm{Z}\right\},$$

as:

$$X_l(x)=A_l e^{i\,l\,x}+C_l,\ l\in \mathrm{Z}, \tag{29}$$

where the undetermined coefficients $A_l,\ C_l$ are to be determined from the loaded differential equation (26). We have:

$$C_l=A_l\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{ilx_m}}{l^2-\bar{\alpha}},\ l\in \mathrm{Z}',\ C_0=A_0.$$

Carrying out the normalization of these coefficients, from this and from (29) we finally obtain the solution of the spectral problem (26):

$$\left\{X_l(x)=e^{i\,l\,x}+\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{ilx_m}}{l^2-\bar{\alpha}},\ \lambda_l^{(1)}=l^2,\ l\in \mathrm{Z}';\ \ X_0(x)=1,\ \lambda_0^{(1)}=\bar{\alpha}\right\}. \tag{30}$$

Relations (24), (28) and (30) imply statement (a) (12) of Lemma 4.1.

(b) Let $\exists\, l_0\in \mathrm{Z}:\ \bar{\alpha}=l_0^2$. By analogy with the previous case (a), we have:

$$C_l=A_l\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{ilx_m}}{l^2-\bar{\alpha}},\ l\in \mathrm{Z}_1'\equiv \mathrm{Z}'\setminus\{\pm l_0\},\ C_0=A_0.$$

Carrying out the normalization of these coefficients, from this and from (29) we finally obtain the system of eigenfunctions and eigenvalues for the spectral problem (26):

$$\left\{X_l(x)=e^{i\,l\,x}+\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{ilx_m}}{l^2-\bar{\alpha}},\ \lambda_l^{(1)}=l^2,\ l\in \mathrm{Z}_1';\right.$$

$$\left.X_{\pm l_0}(x)=1,\ \lambda_{\pm l_0}^{(1)}=\bar{\alpha}=(\pm l_0)^2\right\}. \tag{31}$$

Let us find the associated function for the eigenfunction $X_{\pm l_0}(x) = 1$, corresponding to the eigenvalue $\lambda^{(1)}_{\pm l_0} = \bar{\alpha} = (\pm l_0)^2$. We have:

$$\begin{cases} -\tilde{X}''_{l_0}(x) + \sum\limits_{m=1}^{M} \alpha_m \tilde{X}_{l_0}(x_m) - (l_0)^2 \tilde{X}_{l_0}(x) = (l_0)^2, \\ \tilde{X}^{(j)}_{l_0}(-\pi) = \tilde{X}^{(j)}_{l_0}(\pi), \ j = 0, 1. \end{cases} \tag{32}$$

Hence for the eigenvalue $\sum\limits_{m=1}^{M} \alpha_m = (\pm l_0)^2$ we find the desired associated function:

$$\tilde{X}_{\pm l_0}(x) = e^{\pm i l_0 (x - \bar{x})}. \tag{33}$$

Thus, for the spectral problem (26) the system of eigenfunctions and associated functions and eigenvalues takes the form:

$$\Bigg\{ X_l(x) = e^{i\,l\,x} + \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{i l x_m}}{l^2 - \bar{\alpha}}, \ \lambda^{(1)}_l = l^2, \ l \in \mathrm{Z}'_1; \ X_{\pm l_0}(x) = 1,$$

$$\tilde{X}_{\pm l_0}(x) = e^{\pm i l_0 (x - \bar{x})}, \ \lambda^{(1)}_{\pm l_0} = \bar{\alpha} = (\pm l_0)^2 \Bigg\}. \tag{34}$$

Relations (24), (28) and (34) imply statement (b) (13) of Lemma 4.1. Thus, Lemma 4.1 is completely proved.

6 Proof of Lemma 4.2

Let us find a biorthogonal sequence for (12) (case (a)). We search for it in the form:

$$\{\psi_{kl}(x, y), \ k, \, l \in \mathrm{Z}\} = \left\{ e^{i(lx + ky)}, \ f_0(x)\, e^{iky}, \ l \in \mathrm{Z}', \ k \in \mathrm{Z} \right\}, \tag{35}$$

where only a function $f_0(x)$ is unknown. Using the basis (12), we search for the unknown function $f_0(x)$ in the form:

$$f_0(x) = C_0 + \sum_{r \in \mathrm{Z}'} C_r \left(e^{irx} + \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{ikx_m}}{r^2 - \bar{\alpha}} \right),$$

where $\{C_r, \ r \in \mathrm{Z}'\}$ are unknown coefficients, which must be determined by biorthogonality conditions:

$$(1, \ f_0(x)) = 1; \ \left(e^{ilx} + \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{ilx_m}}{l^2 - \bar{\alpha}}, \ f_0(x) \right) = 0, \ l \in \mathrm{Z}'.$$

The last conditions imply that

$$C_0 = \frac{1}{2\pi} - \sum_{r\in Z'} C_r \cdot \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{irx_m}}{r^2-\bar{\alpha}}, \;\; C_l = -\frac{1}{2\pi} \cdot \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{ilx_m}}{l^2-\bar{\alpha}}, \;\; l\in Z'.$$

Using values C_l, we rewrite C_0:

$$C_0 = \frac{1}{2\pi}\left[1+\sum_{r\in Z'}\left(\frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{irx_m}}{r^2-\bar{\alpha}}\right)^2\right].$$

Further using values C_0, we represent the desired function $f_0(x)$:

$$f_0(x) = -\frac{1}{2\pi} \cdot \sum_{r\in Z} \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{ir(x+x_m)}}{r^2-\bar{\alpha}}. \tag{36}$$

Relations (35) and (36) imply statement (a) (14).

The biorthogonal sequence for (13) (case (b)) is

$$\{\psi_{kl}(x,y),\; k,l\in Z\} = \left\{e^{i(lx+ky)},\; f_0(x)\, e^{iky},\; k\in Z, l\in Z'\right\}, \tag{37}$$

where we have to find the unknown function $f_0(x)$. We search for this function in the form:

$$f_0(x) = C_0 + \sum_{r\in Z'\backslash\{\pm l_0\}} C_r\left(e^{irx} + \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{irx_m}}{r^2-\bar{\alpha}}\right) + C_{l_0}e^{il_0(x-\bar{x})} + C_{-l_0}e^{-il_0(x-\bar{x})},$$

then biorthodonality conditions imply that

$$(1, f_0(x)) = 1; \left(e^{ilx} + \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{irx_m}}{l^2-\bar{\alpha}},\; f_0(x)\right) = 0,\;\; l\in Z'\backslash\{\pm l_0\};$$

$$\left(e^{\pm il_0(x-\bar{x})},\; f_0(x)\right) = 0.$$

First, the conditions $\left(e^{\pm il_0(x-\bar{x})},\; f_0(x)\right) = 0$ imply that $C_{\pm l_0} = 0$,

$$C_0 = \frac{1}{2\pi} - \sum_{r\in Z'\backslash\{\pm l_0\}} C_r \cdot \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{irx_m}}{r^2-\bar{\alpha}}.$$

Further, for $l \in Z' \backslash \{\pm l_0\}$ we find C_l and C_0:

$$C_l = -\frac{1}{2\pi} \cdot \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{ilx_m}}{l^2 - \bar{\alpha}},$$

$$C_0 = \frac{1}{2\pi} \cdot \left[1 + \sum_{r \in Z' \backslash \{\pm l_0\}} \left(\frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{irx_m}}{r^2 - \bar{\alpha}} \right)^2 \right].$$

Furthermore, using the values C_0, we represent the desired function $f_0(x)$:

$$f_0(x) = -\frac{1}{2\pi} \sum_{r \in Z \backslash \{\pm l_0\}} \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{ir(x+x_m)}}{r^2 - \bar{\alpha}}. \tag{38}$$

Relations (37) and (38) imply statement (b) (15). We apply the Paley-Wiener theorem ([14], p.206–207). Thus, Lemma 4.2 is completely proved.

7 Proof of Lemma 4.3

Using the method of separation of variables, and the relationship

$$\frac{-X_l''(x) + \sum\limits_{m=1}^{M} \alpha_m X_l(x_m)}{X_l(x)} + \frac{-Y_k''(y) + \sum\limits_{n=1}^{N} \beta_n Y_k(y_n)}{Y_k(y)} =$$

$$= \lambda_{kl} \equiv \lambda_l^{(1)} + \lambda_k^{(2)}, \;\; k,\, l \in Z, \tag{39}$$

for the solution (11) we obtain the following spectral problems

$$\begin{cases} -Y_k''(y) + \sum\limits_{n=1}^{N} \beta_n Y_k(y_n) = \lambda_k^{(2)} Y_k(y), \;\; k \in Z, \\ Y_k^{(j)}(-\pi) = Y_k^{(j)}(\pi), \;\; j = 0,\, 1, \end{cases} \tag{40}$$

$$\begin{cases} -X_l''(x) + \sum\limits_{m=1}^{M} \alpha_m X_l(x_m) = \lambda_l^{(1)} X_l(x), \;\; l \in Z, \\ X_l^{(j)}(-\pi) = X_l^{(j)}(\pi), \;\; j = 0,\, 1, \end{cases} \tag{41}$$

For problems (40) and (41) we need to consider the following four cases:
(a) $\nexists\, k,\, l \in Z: \;\bar{\alpha} = l^2, \;\bar{\beta} = k^2$,
(b) $\nexists\, k \in Z: \;\bar{\beta} = k^2, \;\exists\, l_0 \in Z: \;\bar{\alpha} = l_0^2$,
(c) $\nexists\, l \in Z: \;\bar{\alpha} = l^2, \;\exists\, k_0 \in Z: \;\bar{\beta} = k_0^2$,
(d) $\exists\, k_0, \;l_0 \in Z: \;\bar{\alpha} = l_0^2, \;\bar{\beta} = k_0^2$.

Since problems (40) and (41) coincide with problem (26), all four cases of Lemma 4.3 are proved in the same way as case (b) of Lemma 4.1.

(a). The case when $\nexists\, k,\, l \in \mathrm{Z}: \ \bar{\alpha} = l^2,\ \bar{\beta} = k^2$. By analogy with case (b) of Lemma 4.1, we obtain the system of eigenfunctions and eigenvalues for spectral problems (40) and (41) respectively in the form

$$\{Y_k(y),\ \lambda_k^{(2)};\ k \in \mathbf{Z}\} = \left\{ e^{iky} + \frac{\sum\limits_{n=1}^{N} \beta_n \cdot e^{iky_n}}{k^2 - \bar{\beta}},\ \ \lambda_k^{(2)} = k^2,\ k \in \mathbf{Z}';\ 1,\ \lambda_0^{(2)} = \bar{\beta} \right\}, \quad (42)$$

$$\{X_l(x),\ \lambda_l^{(1)};\ l \in \mathbf{Z}\} = \left\{ e^{ilx} + \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{ilx_m}}{l^2 - \bar{\alpha}},\ \ \lambda_l^{(1)} = l^2,\ l \in \mathbf{Z}';\ 1,\ \lambda_0^{(1)} = \bar{\alpha} \right\}. \quad (43)$$

Relations (24), (42) and (43) imply statement (a) (16) of Lemma 4.3.

(b). The case when $\nexists\, k \in \mathrm{Z}: \ \bar{\beta} = k^2,\ \exists\, l_0 \in \mathrm{Z}: \ \bar{\alpha} = l_0^2$. By analogy with case (b) of the Lemma 4.1, we obtain a system of eigenfunctions and associated functions and eigenvalues for the spectral problem (41) in the form

$$\{X_l(x),\ \lambda_l^{(1)};\ l \in \mathbf{Z}\} = \left\{ e^{ilx} + \frac{\sum\limits_{m=1}^{M} \alpha_m \cdot e^{ilx_m}}{l^2 - \bar{\alpha}},\ \ \lambda_l^{(1)} = l^2,\ l \in \mathbf{Z}_1'; \right.$$

$$\left. 1,\ \lambda_0^{(1)} = \bar{\alpha},\ e^{\pm i l_0 (x - \bar{x})},\ \lambda_{l_0}^{(1)} = \bar{\alpha} \right\} \quad (44)$$

where the associated functions are the following

$$\{\tilde{X}_{l_0}^{\pm}(x),\ \lambda_{l_0}^{(1)};\ l \in \mathbf{Z}\} = \left\{ e^{\pm i l_0 (x - \bar{x})},\ \lambda_{\pm l_0}^{(1)} = \bar{\alpha} = (\pm l_0)^2 \right\}.$$

Relations (24), (42) and (44) imply statement (b) (17) of Lemma 4.3.

(c). The case when $\nexists\, l \in \mathrm{Z}: \ \bar{\alpha} = l^2,\ \exists\, k_0 \in \mathrm{Z}: \ \bar{\beta} = k_0^2$. By analogy with case (b) of the Lemma 4.1, we obtain a system of eigenfunctions and associated functions and eigenvalues for the spectral problem (40) in the form

$$\{Y_k(y),\ \lambda_k^{(2)};\ k \in \mathbf{Z}\} = \left\{ e^{iky} + \frac{\sum\limits_{n=1}^{N} \beta_n \cdot e^{iky_n}}{k^2 - \bar{\beta}},\ \ \lambda_k^{(2)} = k^2,\ k \in \mathbf{Z}_2'; \right.$$

$$\left. 1,\ \lambda_0^{(2)} = \bar{\beta},\ e^{\pm i k_0 (y - \bar{y})},\ \lambda_{k_0}^{(2)} = \bar{\beta} \right\}. \quad (45)$$

where associated functions are the following

$$\{\tilde{Y}_{k_0}^{\pm}(y),\ \lambda_{k_0}^{(2)};\ k \in \mathbf{Z}\} = \left\{ e^{\pm i k_0 (y - \bar{y})},\ \lambda_{\pm k_0}^{(2)} = \bar{\beta} = (\pm k_0)^2 \right\}.$$

Relation (24), (43) and (45) imply statement (c) (18) of Lemma 4.3.

(d). The case when $\exists\, k_0,\ l_0 \in \mathrm{Z}: \ \bar{\alpha} = l_0^2,\ \bar{\beta} = k_0^2$. In this case, we obtain a system of eigenfunctions and associated functions and eigenvalues for the spectral problems (40) and (41) in the form (44) and (45).

Relation (24), (44) and (45) imply statement (d) (19) of Lemma 4.3. Thus, Lemma 4.3 is completely proved.

8 Proof of Lemma 4.4

Let us find a biorthogonal sequence for (16) (case (a)). We search for it in the form:

$$\{\psi_{kl}(x,y),\ k,l\in \mathrm{Z}\} = \left\{e^{i(lx+ky)},\ g_0(y)e^{ilx},\ f_0(x)e^{iky},\ k,\ l\in \mathrm{Z}',\ f_0(x)g_0(y)\right\}, \quad (46)$$

where only functions $f_0(x),\ g_0(y)$ are unknown. Using the basis (16) by biorthogonality conditions we find the unknown functions $f_0(x),\ g_0(y)$:

$$f_0(x) = -\frac{1}{2\pi}\sum_{r\in \mathrm{Z}}\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{ir(x+x_m)}}{r^2-\bar{\alpha}},\quad g_0(y) = -\frac{1}{2\pi}\sum_{r\in \mathrm{Z}}\frac{\sum\limits_{n=1}^{N}\beta_n\cdot e^{ir(y+y_n)}}{r^2-\bar{\beta}}. \quad (47)$$

Relations (46) and (47) imply statement (a) (20).

Let us find a biorthogonal sequence for (17) (case (b)). We search for it in the form

$$\{\psi_{kl}(x,y),\ k,l\in \mathrm{Z}\} = \left\{e^{i(lx+ky)},\ g_0(y)e^{ilx},\ f_0(x)e^{iky},\ k,\ l\in \mathrm{Z}',\ f_0(x)g_0(y)\right\}, \quad (48)$$

where we have to find the unknown functions $f_0(x),\ g_0(y)$. By appling biorthogonality conditions we have

$$f_0(x) = -\frac{1}{2\pi}\sum_{r\in \mathrm{Z}\setminus\{\pm l_0\}}\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{ir(x+x_m)}}{r^2-\bar{\alpha}},\quad g_0(y) = -\frac{1}{2\pi}\sum_{r\in \mathrm{Z}}\frac{\sum\limits_{n=1}^{N}\beta_n\cdot e^{ir(y+y_n)}}{r^2-\bar{\beta}_n}. \quad (49)$$

Relations (48) and (49) imply statement (b) (21).

Construction of biorthogonal basis for (18) (case (c)) is similar to case (b):

We consider a construction of biorthogonal basis for (19) (case (d)). Let us search it in the form:

$$\{\psi_{kl}(x,y),\ k,l\in \mathrm{Z}\} = \left\{e^{i(lx+ky)},\ g_0(y)e^{ilx},\ f_0(x)e^{iky},\ k,\ l\in \mathrm{Z}',\ f_0(x)g_0(y)\right\}, \quad (50)$$

where it is required to find the unknown functions $f_0(x),\ g_0(y)$. By the orthogonality conditions we have:

$$f_0(x) = -\frac{1}{2\pi}\sum_{r\in \mathrm{Z}\setminus\{\pm l_0\}}\frac{\sum\limits_{m=1}^{M}\alpha_m\cdot e^{ir(x+x_m)}}{r^2-\bar{\alpha}},\quad g_0(y) = -\frac{1}{2\pi}\sum_{r\in \mathrm{Z}\setminus\{\pm k_0\}}\frac{\sum\limits_{n=1}^{N}\beta_n\cdot e^{ir(y+y_n)}}{r^2-\bar{\beta}}. \quad (51)$$

Relations (50) and (51) imply statement (d) (23). We apply the Paley-Wiener theorem ([14], p.206–207). Thus, Lemma 4.4 is completely proved.

9 Algorithm for solving stabilization problem

We propose the following algorithm for solving the stabilization problem for the heat equation with a loaded two-dimensional Laplace operator. It consists of the following constructively implemented steps.

Step 1. We define the function $z_0(x,y)$ as a continuation of the given function $u_0(x,y)$. Thus in the auxiliary boundary value problem (5)–(7) it is needed to continue the function $z_0(x,y)$ on the square Ω_1, so that the requirement (8) is satisfied for a solution $z(x,y,t)$ of the problem (5)–(7). In this case the condition (4) holds as well for its restriction $u(x,y,t)$ and a required boundary control $p(x,y,t)$, $\{x,y\} \in \Sigma$ is defined as a trace of the function $z(x,y,t)$ for $\{x,y,t\} \in \Sigma$.

Step 2. We construct complete biorthogonal system of functions on the square Ω_1 by solving appropriate spectral problems.

Step 3. Find the coefficients of the decomposition for the desired function $z_0(x,y)$ on the square Ω_1 from constructed at the previous step complete biorthogonal system so that the condition (8) holds.

Step 4. By the found solution $z(x,y,t)$ of the auxiliary boundary value problem (5)–(7) as restriction of it to the cylinder Q we find a solution $u(x,y,t)$ to the given boundary value problem (1)–(3), satisfying the required condition (4). A boundary control $p(x,y,t)$, $\{x,y\} \in \Sigma$ is found as a trace of the solution $z(x,y,t)$, i.e.

$$p(x,y,t) = z(x,y,t)_{|\{x,y,t\}\in\Sigma}.$$

10 Conclusion

The results of the work on the spectral properties of a loaded two-dimensional Laplace operator can be useful in solving stabilization problems for a loaded parabolic equation with the help of boundary control actions that can be used in problems of mathematical modeling by controlled loaded differential equations.

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